Closing Today: HW_1A,1B(4.9,5.1)
Closing Fri: HW_1C(5.2)
Closing next Wed: HW_2A,2B,2C(5.3-5)
Office Hours: 2:00-3:30, MSC-Com B-014

5.3 Fundamental Theorem of Calculus (FTOC)

Consider *v(t) = 3t* feet/sec

Entry Task: Find the areas (from graph)

1. $\int_{0}^{2} v(t)dt$

2. $\int_{0}^{10} v(t)dt$

3. $d(x) = \int_{0}^{x} v(t)dt$

Any observations?

What changes if the lower bound a number other than 0?

For example, let's try 4?

$$4.\ h(x) = \int_4^x v(t)dt$$

Fundamental Theorem of Calculus (Part 1):
Accumulated Area functions under graphs are antiderivatives!

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

That is, for any constant a, the "accumulated signed area" formula

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of $f(x)$.

Proof sketch: If $g(x) = \int_a^x f(t)dt$
then, by definition,

$$g'(x) = \lim_{h \to o} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to o} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$

$$= \lim_{h \to o} \frac{\int_x^{x+h} f(t)dt}{h}.$$

As $h \to 0$, $\int_x^{x+h} f(t)dt \to f(x)h$
(*assuming continuity*).

So $g'(x) = \lim_{h \to o} \frac{f(x)h}{h} = f(x)$

Mechanically using FTOC (Part 1)

Compute the **derivatives** of the following functions:

1. $g(x) = \int_{3}^{x} \cos(t)\, dt$

2. $h(x) = \int_{x}^{-2} te^{t} dt$

3. $f(x) = \int_{0}^{x^3} t + \sin(t)\, dt$

4. $k(x) = \int_{1+x^2}^{x^3} \sqrt{2+t}\, dt$

General form of FTOC (Part 1):

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)dt\right) = f\big(h(x)\big)h'(x) - f\big(g(x)\big)g'(x)$$

Fundamental Theorem of Calculus (Part 2):

If F(x) <u>any</u> antiderivative of f(x),

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof sketch:

Let $F(x)$ by any antiderivative.

By the FTOC (Part 1), we can think of $F(x)$ as an accumulated area function such as

$$F(x) = \int_c^x f(t)dt$$

for some number *c*.

Then using integral (area) rules we have

$$F(b) - F(a)$$

$$= \int_c^b f(t)dt - \int_c^a f(t)dt$$

$$= \int_c^b f(t)dt + \int_a^c f(t)dt$$

$$= \int_a^b f(t)dt$$

Mechanically using FTOC (Part 2)

Evaluate

1. $\int_0^1 x^3 \, dx$

2. $\int_0^{\pi} \sin(t) \, dt$

3. $\int_{1}^{4} \frac{1}{w}\,dw$

5. $\int_{1}^{2} \frac{3}{x^2}\,dx$

4. $\int_{0}^{5} e^x\,dx$

6. $\int_{1}^{4} \sqrt{x}\,dx$

7. $\int_0^1 \frac{1}{1+x^2}\,dx$

8. $\int_0^{\pi/3} \sec(x)\tan(x)\,dx$

5.4 The Indefinite Integral and Net/Total Change

Def'n: The **indefinite integral** of $f(x)$ is defined to be the general antiderivative of $f(x)$.
And we write

$$\int f(x)dx = F(x) + C,$$

where $F(x)$ is any antiderivative of $f(x)$.

Example:

$$\int 6e^x + 4x - 5\sqrt{x}\,dx$$

Net Change and Total Change

The FTOC(2) says the **net change** in $f(x)$ from $x = a$ to $x = b$ is the integral of its **rate**. That is:

$$\int_a^b f'(t)dt = f(b) - f(a)$$

For example

Assume an object is moving along a straight line (up/down or left/right).

$s(t)$ = 'location at time t'
$v(t)$ = 'velocity at time t'

pos. $v(t)$ means moving up/right
neg. $v(t)$ means moving down/left

The FTOC (part 2) says

$$\int_a^b v(t)dt = s(b) - s(a)$$

i.e.

'integral of velocity'= '**net change** in dist'

We also call this the *displacement*.

We define **total change** in dist. by

$$\int_{a}^{b} |v(t)| dt$$

which we compute by

1. Solving $v(t) = 0$ for t.

2. Splitting up the integral at these t values. Then dropping the absolute value and integrating separately.

3. Adding together as positive numbers.

Example: $v(t) = t^2 - 2t - 8$ ft/sec
Compute the total distance traveled from t = 1 to t = 6.

A brief pause to discuss current integration methods. We can currently find antiderivatives for sums and constant multiples of functions ***directly*** from our integration table.

Examples (we **can** currently do):

1. $\int 6e^x + 4x - 5\sqrt{x}\, dx$

2. $\int 6\sec^2(\mathrm{x}) - \frac{9}{x^4}\, dx$

Examples we **cannot** currently do (but will be able to do later in the term):

$$\int xe^{3x}dx\,; \qquad \int \tan(x)dx$$

$$\int x\sin(x^2)\,dx; \qquad \int \frac{x^2 - x + 6}{x^3 + 3x}dx$$

$$\int \frac{3}{x - 2\sqrt{x}}dx\,; \qquad \int \frac{\sqrt{x^2 - 1}}{x^2}dx$$

Examples we will "*never*" be able to do:

$$\int \mathrm{e}^{\mathrm{x}^2}dx\,; \int \sec(x^2)dx$$

Here are two that look bad but we can currently do them, why?

1. $\displaystyle\int \frac{\sqrt{\mathrm{x}} - 3\mathrm{x}}{\mathrm{x}}dx$

2. $\displaystyle\int \frac{\cos(x)}{1 - \cos^2(x)}dx$

What is the value of:

$$\int_{\pi/4}^{\pi/2} \frac{\cos(x)}{1 - \cos^2(x)}\, dx$$